Localization and hybridization improve the accuracy of ensemble-based covariances. Localization functions and hybridization weights can be jointly and objectively optimized. The proposed method uses the ensemble members only and is affordable for high-dimensional systems. It has been tested on various atmospheric and oceanographic models (ARPEGE/AROME, GFS, MPAS, WRF, NEMO). Localization and hybridization diagnostics can be used for both EnVar algorithms and sequential filters (e.g. EnKF).

### Theory

- Covariance matrix sampled from $N$ members: $\mathbf{B}^*$
- Asymptotic value for $N \to \infty$: $\mathbf{B}$
- $4^{th}$ order centered moment sampled from $N$ members: $\mathcal{E}

\[ \mathbb{E} \left[ \mathbf{B}_{ij}^2 \right] = P(N) \mathbb{E} \left[ \mathbf{B}_i^2 \right] + Q(N) \mathbb{E} \left[ \mathbf{B}_i \mathbf{B}_j \right] + R(N) \mathbb{E} \left[ \mathbf{B}_j \mathbf{B}_i \right] \]

where $P$, $Q$ and $R$ are known fractions of polynomials.

- Localized covariance matrix: $\mathbf{\hat{B}}_l = \mathbf{L} \circ \mathbf{B}$
- Optimal localization matrix $\mathbf{L}$ minimizes $\mathbb{E} \left[ \left\| \mathbf{B} \circ \mathbf{L} \circ \mathbf{B} \right\|^2 \right]

\[
\mathbf{L}_l = \frac{\mathbb{E} \left[ \mathbf{B}_{ij}^2 \right]}{\mathbb{E} \left[ \mathbf{B}_{ii}^2 \right]}
\]

- Static covariance matrix for hybridization: $\mathbf{\hat{B}}$
- Localized/hybridized covariance matrix: $\mathbf{\hat{B}}^h = \mathbf{L}^h \circ \mathbf{B} + \beta c^2 \mathbf{\hat{B}}$
- Optimal $\mathbf{L}^h$ and static weight $\beta c^2$ minimize $\mathbb{E} \left[ \left\| \mathbf{B}^h \circ \mathbf{L}^h \circ \mathbf{B} \right\|^2 \right]

\[
\sum_{ij} \left( 1 - \mathbf{L}_{ij} \right) \mathbb{E} \left[ \mathbf{B}_i \mathbf{B}_j \right] \quad \text{and} \quad \mathbf{L}^h_l = \frac{\mathbb{E} \left[ \mathbf{B}_{ij}^2 \right]}{\mathbb{E} \left[ \mathbf{B}_{ii}^2 \right]} \beta c^2 \mathbf{B}_{jj}
\]


### Implementation

- Expectations $\mathbb{E}[\cdot]$ estimated via an ergodicity assumption.
- For instance, spatial and angular ergodicity: quantities are sampled with couples of points for each separation class.

### Summary

- Localization and hybridization improve the accuracy of ensemble-based covariances.
- Localization functions and hybridization weights can be jointly and objectively optimized.
- The proposed method uses the ensemble members only and is affordable for high-dimensional systems.
- It has been tested on various atmospheric and oceanographic models (ARPEGE/AROME, GFS, MPAS, WRF, NEMO).
- Localization and hybridization diagnostics can be used for both EnVar algorithms and sequential filters (e.g. EnKF).

### Results for the ARPEGE EDA, mid-troposphere temperature

#### Localization only

- Localization length-scale and amplitude increase with increasing $N$ (less sampling noise to filter).
- Localization top is flatter than the correlation top.

#### Localization and hybridization

- If $N$ increases: more weight on $\mathbf{L}^h \circ \mathbf{B}$, less on $\mathbf{B}$.
- $\mathbf{L}^h_l + \beta c^2$ can be different from 1 depending on $\mathbf{B}_i$ and $\mathbf{B}_j$ (if $\mathbf{B}_i$ were twice larger, $\beta c^2$ would be twice smaller).

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An open-source and generic code is available at: https://opensource.cnrm-game-meteo.fr/projects/hybrid_diag
Contact: benjamin.menetrier@meteo.fr